Continuous discrete observer with updated sampling period (long version)

Vincent Andrieu, Madiha Nadri, Ulysse Serres, Jean-Claude Vivalda

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Vincent Andrieu [a,c] Madiha Nadri [a] Ulysse Serres [a] Jean-Claude Vivalda [b]

[a] *Université de Lyon, F-69622, Lyon, France; Université Lyon 1, Villeurbanne; CNRS, UMR 5007, LAGEP, 43 bd du 11 novembre, 69100 Villeurbanne, France.*

https://sites.google.com/site/vincentandrieu/, nadri@lagep.univ-lyon1.fr, ulysse.serres@univ-lyon1.fr.

[b] *Inria, CORIDA, Villers-lès-Nancy, F- 54600, France*
Université de Lorraine , IECL, UMR 7502, Vandœuvre-lès-Nancy, F-54506, France
CNRS, IECL,UMR 7502, Vandœuvre-lès-Nancy, F-54506, France

e-mail: jean-claude.vivalda@inria.fr

[c] *Fachbereich C - Mathematik und Naturwissenschaften, Bergische Universitat Wuppertal, Gaustrae 20, 42097 Wuppertal, Germany.*

Abstract

This paper deals with the design of high gain observers for a class of continuous-time dynamical systems with discrete-time measurements. Indeed, different approaches based on high gain techniques have been followed in the literature to tackle this problem. Contrary to these works, the measurement sampling time is considered to be variable. Moreover, the new idea of the proposed work is to synthesize an observer requiring the less knowledge as possible from the output measurements. This is done by using an updated sampling time observer. The vector fields related to the systems considered in this paper are assumed to be globally Lipschitz. Under this assumption, the asymptotic convergence of the observation error is established. As an application of this approach, a state estimation problem of an academic bioprocess is studied, and its simulation results are discussed.

Key words: Nonlinear systems, sampled-data, continuous-discrete time observers, high gain observer, updated sampling-time.

1 Introduction

Estimating the state of a partially measured dynamical system is a classical problem in control theory. An algorithm that solves this problem is an asymptotically convergent observer. When the measurement is available only at some discrete-time instant, a continuous-discrete time observer has to be designed. The study of this type of algorithm can be traced back to Jazwinski who introduced the continuous-discrete Kalman filter to solve a filtering problem for stochastic continuous-discrete time systems (see [10]). Inspired by this approach, the continuous discrete high-gain observer has been studied in [7]. Since then, different approaches have been investigated. The robustness of an observer with respect to time discretization was studied in [5] (see also [16]). In [14], a Newton observer is provided which estimates the state at time t_k from N consecutive measurements of outputs and inputs; in [6], the authors show how this method can be implemented in the case where the sampled system is not known analytically. In [11] observers were designed from an output predictor (see also related works in [1]). Some other approaches based on time delayed techniques have also been considered in [18]. Recently, a new continuous-discrete observer design methodology for Lipschitz nonlinear systems based on reachability analysis was presented in [8].

In this note, we consider also the design of a continuous discrete time observer. However, in opposition to these results, we consider the case in which the sampling time is variable and used as a tuning parameter. More precisely, we consider that the quantity $t_{k+1}-t_k$ is a part of the design of the continuous discrete observer. Hence, in the proposed algorithm, the measurement time is computed online. In fact, the use of sensors follows an event

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based on an extended observer state component. This may be related to the event-triggered control methodology (see for instance [19,20]).

In high-gain designs, the asymptotic convergence of the estimate to the state is obtained by dominating the Lipschitz nonlinearities with high-gain techniques. However, there is a trade-off between the high-gain parameter and the measurement step size and can lead to restrictive design conditions on the sampling measurement time (see also [15]). Inspired by [4], the extra observer state component estimates the local Lipschitz constant in order to maximize the measurement stepsize.

The paper is organized as follows. The class of systems considered and the structure of the estimation algorithm are given in Section 2. The main result and its proof are given in section 3. Section 4 contains an illustrative example. Finally, Section 5 is devoted to the conclusion.

2 Problem statement and structure of the observer

2.1 Class of systems considered

In this work we consider the problem of designing an observer for nonlinear systems that are diffeomorphic to the following form

$$\dot{x} = Ax + f(x,u), \tag{1}$$

where the state x is in $\mathbb{R}^n$; $u : \mathbb{R} \to \mathbb{R}^p$ is a known input in $\mathbb{L}^\infty(\mathbb{R}_+, \mathbb{R}^p)$, A is a matrix in $\mathbb{R}^{n\times n}$ and $f : \mathbb{R}^n \times \mathbb{R}^p$ is a locally Lipschitz vector field both having the following triangular structure :

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}, \quad f(x,u) = \begin{bmatrix} f_1(x_1,u) \\ f_2(x_1,x_2,u) \\ \vdots \\ f_n(x,u) \end{bmatrix}.$$

The measured output is given as a sequence of values $(y_k)_{k\geqslant 0}$ in $\mathbb{R}$

$$y_k = Cx(t_k), \tag{2}$$

where $(t_k)_{k\geqslant 0}$ is a sequence of times to be selected and $C = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}$ is in $\mathbb{R}^n$. In this paper, we shall denote by $\langle \cdot, \cdot \rangle$ the canonical scalar product in $\mathbb{R}^n$ and by $\|\cdot\|$ the related Euclidean norm; we shall use the same notation for the corresponding induced matrix norm. Also, we use the symbol $'$ to denote the transposition operation.

We consider the case in which the vector field $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ satisfies the following assumption.

Assumption 1 *The function $f = (f_1, \dots, f_n)'$ is such that the following incremental bound is satisfied for all $(x, e, u) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^p$,*

$$|f_j(x+e,u) - f_j(x,u)| \leqslant c(x,u) \sum_{i=1}^{j} |e_i| \ , \tag{3}$$

where $c : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}_+$ is a continuous function which satisfies the following bound

$$c(x,u) \leqslant \Gamma(u) \ , \ \forall \ (x,u) \in \mathbb{R}^n \times \mathbb{R}^p \ , \tag{4}$$

where $\Gamma : \mathbb{R}^p \to \mathbb{R}_+$.

Compared to the preliminary version of this work presented in [2], now a larger class of nonlinear system is addressed. Indeed, general upper triangular systems are now allowed.

Note that in the case in which we know a bound on the input u then this would imply that we come back to the globally Lipschitz context. However, even in this case, we believe that employing a tighter bound in term of a state-dependent function c implies that the sensors are less used than they would be if we were considering directly the Lipschitz bound.

2.2 Updated sampling time observer

The continuous-discrete time observer with updated sampling period is given by [1]

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + f(\hat{x}(t), u(t)), \quad \forall \ t \in [t_k, t_{k+1}) \ , \\ \hat{x}(t_k) = \hat{x}(t_k^-) + \delta_k \mathcal{L}(t_k^-) K (C\hat{x}(t_k^-) - y_k), \end{cases} \tag{5}$$

where K is a gain matrix. The matrix function $\mathcal{L} : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ is defined as $\mathcal{L}(t) = \mathrm{diag}(L(t), \dots, L(t)^n)$ with $L : \mathbb{R}_+ \to \mathbb{R}$ is given as a solution to the following system of continuous discrete differential equations

$$\begin{aligned} \dot{L}(t) &= a_2 L(t) M(t) c(\hat{x}(t), u(t)), \quad t \in [t_{k-1}, t_k) && \text{(6a)} \\ \dot{M}(t) &= a_3 M(t) c(\hat{x}(t), u(t)), \quad t \in [t_{k-1}, t_k) && \text{(6b)} \\ L(t_k) &= L(t_k^-)(1 - a_1\alpha) + a_1\alpha && \text{(6c)} \\ M(t_k) &= 1, && \text{(6d)} \end{aligned}$$

initiated from $L(0) \geqslant 1$ and with $a_1\alpha < 1$. We have for all k,

$$y_k = Cx(t_k),$$

where the t_k's, k in $\mathbb{N}$ are given by the following relations,

$$t_0 = 0, \ t_{k+1} = t_k + \delta_k,$$

[1] The solution $\hat{x}(\cdot)$ is a right-continuous function. Given a right-continuous function $\phi : \mathbb{R} \to \mathbb{R}^n$, the notation $\phi(t^-)$ stands for $\phi(t^-) = \lim_{h\to 0, h<0} \phi(t+h)$.

$$\delta_k = \min\{s \in \mathbb{R}_+ \mid sL((t_k + s)^-) = \alpha\}, \qquad (7)$$

where α, a_1, a_2 and a_3 are positive real numbers to be chosen.

2.3 About the updating time period

To understand the motivation in this update law note that a first order approximation gives

$$L(t_{k+1}^-) = L(t_k) + a_2 L(t_k) c(\hat{x}(t_k), u(t_k))\delta_k + o(\delta_k) \ .$$

Hence, it yields,

$$\frac{L(t_{k+1}) - L(t_k)}{\delta_k} = L(t_k)\left[a_1(1 - L(t_k)) + a_2 c(\hat{x}(t_k), u(t_k))\right] + o(1) \ .$$

We recognize here the same update law structure than the one introduce in [17] which was motivated by a Riccati equation.

Note that for all k, δ_k is well defined. Indeed, L is not decreasing in every time interval $[t_k, t_{k+1})$. Moreover, when there is a jump (i.e., when there exists k such that $t = t_k$), we see that $L(t_k) \geqslant 1$ if $L(t_k^-) \geqslant 1$. Hence, we get $L(t) \geqslant 1$ for every $t \geqslant 0$. The function $s \mapsto sL(t_k + s)$ being continuous, zero at zero and going to infinity (if there is no jump), the existence of δ_k is well defined by (7). Also, we have $\delta_k < \alpha$ for all k.

Moreover, we have the following lemma which shows that if the input is bounded then the high-gain parameter L is bounded along solutions.

Lemma 1 (Boundedness of L) *If u is in $\mathbb{L}^\infty(\mathbb{R}_+, \mathbb{R}^p)$ then there exists ℓ_∞ (depending on the initial conditions for system (1) and its observer (5)-(6)) such that $1 \leqslant L(t) \leqslant \ell_\infty$ for every $t \geqslant 0$.*

The proof of Lemma 1 is postponed to Appendix A.1. Note that δ_k is lower bounded since, by definition, we have $L(t_{k+1}^-)\delta_k = \alpha$. These comments imply that for all essentially bounded input time function u, the sequence of sampling period $(\delta_k)_{k\in\mathbb{N}}$ is well defined, upper and lower bounded, for all k in $\mathbb{N}$ and that $\lim_{k\to+\infty} t_k = +\infty$. Note that if we know a bound on u, the function $c(\hat{x}, u)$ in (6) could simply be replaced by a constant depending on the function Γ. Note however that in this case, $\dot{L}$ is larger and this reduces the size of each sampling period $(\delta_k)_{k\in\mathbb{N}}$. Consequently, the sensors are more frequently employed which is something we would like to avoid.

3 Observer convergence

With the property given above in hand, we are now able to state our main result.

Theorem 2 (Updating continuous-discrete time observer) *There exist a gain matrix K and $\alpha_m > 0$ such that for every α in $(0, \alpha_m]$, there exist positive numbers a_1, a_2 and a_3 such that*[2] *for every essentially bounded input functions the estimation error obtained using the observer (5)-(6) converges asymptotically toward zero. More precisely, for every initial condition $(x(0), \hat{x}(0))$ in $\mathbb{R}^n \times \mathbb{R}^n$ and $L(0)) \geqslant 1$, for every input function u in $\mathbb{L}^\infty(\mathbb{R}_+, \mathbb{R}^p)$ the associated solution to system (1)-(5)-(6) satisfies* $\lim_{t\to+\infty} \|x(t) - \hat{x}(t)\| = 0$.

Proof. Let D be the diagonal matrix in $\mathbb{R}^{n\times n}$ defined by $D = \mathrm{diag}(1, 2, \ldots, n)$. Let P be a symmetric positive definite matrix in $\mathbb{R}^{n\times n}$ and K a vector in $\mathbb{R}^n$ such that the following inequality is satisfied (see [17, equation (14)] or [13, equation (18)] or [3])

$$p_1 I \leqslant P \leqslant p_2 I \ , \qquad (8)$$

I being the identity matrix, and

$$\begin{aligned} (A + KC)'P + P(A + KC) &\leqslant -I \ , \\ p_3 P \leqslant PD + DP &\leqslant p_4 P \ , \end{aligned} \qquad (9)$$

with $p_1, \ldots, p_4$ positive real numbers. Let $e \triangleq \hat{x} - x$ be the estimation error; e satisfies the following differential equation (*cf.* equations (1)-(5)) for all $t \in [t_k, t_{k+1})$

$$\dot{e}(t) = Ae(t) + \Delta(\hat{x}(t), e(t), u(t)), \qquad (10)$$

where the function $\Delta : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is defined as

$$\Delta(\hat{x}, e, u) = f(\hat{x}, u) - f(\hat{x} - e, u) \ , \ \forall \ (\hat{x}, e, u) \ .$$

From Assumption 1 (i.e., inequality (3)), this function satisfies $|\Delta_j(\hat{x}, e, u)| \leqslant c(\hat{x}, u) \sum_{i=1}^{j} |e_i|$ for all $(\hat{x}, e, u)$.

In the sequel, and using the results presented in [12] (see also [4]) we consider the scaled observation error defined for all t by $E(t) = \mathcal{L}(t)^{-1} e(t)$. Also, to simplify the presentation, we introduce the notations

$$\begin{aligned} L_k^- &= L(t_k^-), & \mathcal{L}_k^- &= \mathcal{L}(t_k^-), & L_k &= L(t_k), \\ \mathcal{L}_k &= \mathcal{L}(t_k), & E_k &= E(t_k) & e_k &= e(t_k) \,. \end{aligned}$$

If we integrate equation (10) on the interval $[t_k, t_k + \tau)$ with $\tau < \delta_k$, we get

$$e(t_k + \tau) = \exp(A\tau)e(t_k) + \int_0^\tau \exp(A(\tau - s))\Delta(\hat{x}(t_k + s), e(t_k + s), u(t_k + s))\mathrm{d}s \qquad (11)$$

[2] In the design, K is selected in (8)-(9). Then we set $a_1 = \frac{1}{2p_2p_4}$. α_m is selected sufficiently small such that $a_1\alpha_m < 1$ and (17) holds. Finally, we set $a_3 = 2n$ and $a_2 \geqslant 2\frac{N_1(\alpha)+N_2(\alpha)}{p_3p_1}$ where $\alpha \leqslant \alpha_m$ and N_1 and N_2 are given in the proof of Lemma 5.

Moreover, from (5), we get

$$e_{k+1} = (I + \delta_k \mathcal{L}(t_{k+1}^-)KC)e((t_k+\delta_k)^-). \tag{12}$$

In the remaining part of the proof, we shall show that the Lyapunov function $V(E(t_k)) = E(t_k)'PE(t_k)$ is decreasing toward zero along the solution to the system. In order to evaluate the Lyapunov function, let us first mention the following algebraic properties of the matrix function $\mathcal{L}_k$:

$$\begin{aligned}\left(\mathcal{L}_{k+1}^-\right)^{-1}(I+\delta_k\mathcal{L}_{k+1}^-KC) &= (I+\delta_k L_{k+1}^-KC)\left(\mathcal{L}_{k+1}^-\right)^{-1}\\ &= (I+\alpha KC)\left(\mathcal{L}_{k+1}^-\right)^{-1},\end{aligned} \tag{13}$$

where the last equality has been obtained from (7). Moreover, since for all k, $\left(\mathcal{L}_k^-\right)^{-1}A = L_k^- A\left(\mathcal{L}_k^-\right)^{-1}$, it yields for all k and all $i \geqslant 1$

$$\left(\mathcal{L}_k^-\right)^{-1}A^i = L_k^- A\left(\mathcal{L}_k^-\right)^{-1}A^{i-1} = (L_k^- A)^i\left(\mathcal{L}_k^-\right)^{-1},$$

and

$$\begin{aligned}\left(\mathcal{L}_k^-\right)^{-1}\exp(As) &= \left(\mathcal{L}_k^-\right)^{-1}\sum_{i=0}^{+\infty}\frac{A^i s^i}{i!}\\ &= \exp(L_k^- As)\left(\mathcal{L}_k^-\right)^{-1}.\end{aligned} \tag{14}$$

Hence, employing the previous algebraic equalities (13) and (14), together with relation (11), we get, when left multiplying (12) by $\left(\mathcal{L}_{k+1}^-\right)^{-1}$,

$$\left(\mathcal{L}_{k+1}^-\right)^{-1}e_{k+1} = Q(\alpha)(\mathcal{L}_{k+1}^-)^{-1}\mathcal{L}_k E_k + R,$$

with

$$Q(\alpha) = (I+\alpha KC)\exp(A\alpha),$$

and

$$\begin{aligned}R = (I+\alpha KC)\int_0^{\delta_k}&\exp(AL_{k+1}^-(\delta_k-s))\left(\mathcal{L}_{k+1}^-\right)^{-1}\\ &\cdot\Delta(\hat{x}(t_k+s), e(t_k+s), u(t_k+s))ds.\end{aligned} \tag{15}$$

Note that, since we have $E_{k+1} = \Psi\left(\mathcal{L}_{k+1}^-\right)^{-1}e_{k+1}$ with $\Psi = (\mathcal{L}_{k+1})^{-1}\mathcal{L}_{k+1}^-$, it yields

$$V(E_{k+1}) = V(E_k) + T_1 + T_2,$$

with

$$\begin{aligned}T_1 = E_k'\,\mathcal{L}_k(\mathcal{L}_{k+1}^-)^{-1}Q(\alpha)'\,\Psi P\Psi\,Q(\alpha)\,(\mathcal{L}_{k+1}^-)^{-1}\mathcal{L}_k\,E_k&\\ -V(E_k),&\end{aligned}$$

$$T_2 = 2\,E_k'\,(\mathcal{L}_{k+1})^{-1}\mathcal{L}_k Q(\alpha)'\Psi P\Psi R + R'\Psi P\Psi R.$$

The remaining part of the proof is divided into three parts. The first two ones are devoted to upper bound the two terms T_1 and T_2 and the last one is devoted to the Lyapunov analysis. The fact that the Lyapunov function is decreasing is due to the term T_1 which will be shown to be negative. The second term is handled by robustness.

Step 1 : Upper bounding T_1

Lemma 3 *Let $a_1 = \frac{1}{2p_2p_4}$. There exists $\alpha_m > 0$ sufficiently small such that for all α in $[0,\alpha_m)$*

$$\begin{aligned}T_1 \leqslant -\left(\frac{a_2p_3p_1}{a_3}\left[\exp\left(a_3\int_0^{\delta_k}c(r)dr\right)-1\right]+\frac{\alpha p_1}{4p_2}\right)&\\ \times\left\|\left(\mathcal{L}_{k+1}^-\right)^{-1}e_k\right\|^2,&\end{aligned} \tag{16}$$

where $c(r) = c(\hat{x}(r), u(r))$.

The proof of Lemma 3 uses the following lemma whose proof is given in Appendix.

Lemma 4 *Taking a_1 sufficiently small, there exists $\alpha_m > 0$ sufficiently small such that for all $\alpha < \alpha_m$ we have*

$$Q(\alpha)'\Psi P\Psi Q(\alpha) \leqslant P - \alpha\frac{1}{4p_2}P. \tag{17}$$

Proof of Lemma 3. We have, for all v in $\mathbb{R}^n$

$$\begin{aligned}&v'\mathcal{L}_k\left(\mathcal{L}_{k+1}^-\right)^{-1}P\left(\mathcal{L}_{k+1}^-\right)^{-1}\mathcal{L}_k v - v'Pv\\ &\quad= v'\left(\int_{t_k}^{t_{k+1}}\mathcal{L}_k\frac{d}{ds}\left(\mathcal{L}(s)^{-1}\right)P\mathcal{L}(s)^{-1}\mathcal{L}_k\right.\\ &\qquad\qquad\left.+\mathcal{L}_k\mathcal{L}(s)^{-1}P\frac{d}{ds}\left(\mathcal{L}(s)^{-1}\right)\mathcal{L}_k ds\right)v.\end{aligned}$$

However, we have for all s in $[t_k, t_{k+1})$

$$\frac{d}{ds}\left(\mathcal{L}(s)^{-1}\right) = -\frac{\dot{L}(s)}{L(s)}D\mathcal{L}(s)^{-1}.$$

Consequently, it yields

$$\begin{aligned}&v'\mathcal{L}_k\left(\mathcal{L}_{k+1}^-\right)^{-1}P\left(\mathcal{L}_{k+1}^-\right)^{-1}\mathcal{L}_k v - v'Pv\\ &= -v'\left(\int_{t_k}^{t_{k+1}}\frac{\dot{L}(s)}{L(s)}\mathcal{L}_k\mathcal{L}(s)^{-1}[PD+DP]\mathcal{L}(s)^{-1}\mathcal{L}_k ds\right)v.\end{aligned}$$

Bearing in mind that $L \geqslant 1$ and $\dot{L} \geqslant 0$ and taking into account the bounds on $DP + PD$ in (9) and P in (8), we get

$$
\begin{aligned}
& v'\mathcal{L}_k\left(\mathcal{L}_{k+1}^{-}\right)^{-1} P \left(\mathcal{L}_{k+1}^{-}\right)^{-1}\mathcal{L}_k v - v'Pv \\
& \quad \leqslant -p_3 v'\Bigg(\int_{t_k}^{t_{k+1}} a_2 c(s)\exp\left(a_3\int_{t_k}^{s} c(r)dr\right) \\
& \qquad\qquad \mathcal{L}_k\mathcal{L}(s)^{-1}P\mathcal{L}(s)^{-1}\mathcal{L}_k ds\Bigg)v \\
& \quad \leqslant -p_3 p_1 v'\Bigg(\int_{t_k}^{t_{k+1}} a_2 c(s)\exp\left(a_3\int_{t_k}^{s} c(r)dr\right) \\
& \qquad\qquad \mathcal{L}_k\mathcal{L}(s)^{-1}\mathcal{L}(s)^{-1}\mathcal{L}_k ds\Bigg)v.
\end{aligned}
$$

Note that since $L_k \leqslant L(s) \leqslant L_{k+1}^{-}$, we finally get

$$
\begin{aligned}
& v'\mathcal{L}_k\left(\mathcal{L}_{k+1}^{-}\right)^{-1} P \left(\mathcal{L}_{k+1}^{-}\right)^{-1}\mathcal{L}_k v - v'Pv \\
& \quad \leqslant -\frac{a_2 p_3 p_1}{a_3}\left[\exp\left(a_3\int_{t_k}^{t_{k+1}} c(r)dr\right) - 1\right]\times \\
& \qquad\qquad \left\|\left(\mathcal{L}_{k+1}^{-}\right)^{-1}\mathcal{L}_k v\right\|^2.
\end{aligned}
$$

Consequently, the bound (16) is obtained from the previous inequality with $v = E_k$, and from inequality (17) in Lemma 4 together with (8).

Step 2 : Upper bounding T_2

Lemma 5 *There exist two continuous functions N_1 and N_2 such that the following inequality holds*

$$
\begin{aligned}
T_2 \leqslant & \left\|\left(\mathcal{L}_{k+1}^{-}\right)^{-1} e_k\right\|^2 \times \qquad (18) \\
& \left[N_1(\alpha)\left[\exp\left(\int_0^{\delta_k} nc(t_k+r)dr\right) - 1\right]\right. \\
& \left. + N_2(\alpha)\left[\exp\left(n\int_0^{\delta_k} c(t_k+r)dr\right) - 1\right]^2\right].
\end{aligned}
$$

Proof. In order to prove inequality (18), let us first analyze the term R given by equation (15). First, we seek for an upper bound of the norm of $\left(\mathcal{L}_{k+1}^{-}\right)^{-1}\Delta(\hat{x}(t_k + s), e(t_k+s), u(t_k+s))$, we have

$$
\begin{aligned}
& \left\|\left(\mathcal{L}_{k+1}^{-}\right)^{-1}\Delta(\hat{x}(t_k+s), e(t_k+s), u(t_k+s))\right\| \\
& \leqslant \left(\sum_{j=1}^{n}\left(L_{k+1}^{-}\right)^{-2j} c^2(t_k+s)\left(\sum_{i=1}^{j}|e_i(t_k+s)|\right)^2\right)^{1/2} \\
& = \left(\sum_{j=1}^{n} c^2(t_k+s)\left(\sum_{i=1}^{j}\left(L_{k+1}^{-}\right)^{-j}|e_i(t_k+s)|\right)^2\right)^{1/2}.
\end{aligned}
$$

Since, $L_{k+1} \geqslant 1$, it yields

$$
\begin{aligned}
& \left\|\left(\mathcal{L}_{k+1}^{-}\right)^{-1}\Delta(\hat{x}(t_k+s), e(t_k+s), u(t_k+s))\right\| \\
& \leqslant \left(\sum_{j=1}^{n} c^2(t_k+s)\left(\sum_{i=1}^{j}\left(L_{k+1}^{-}\right)^{-j}|e_i(t_k+s)|\right)^2\right)^{1/2} \\
& \leqslant \left(\sum_{j=1}^{n} c^2(t_k+s)\left(\sum_{i=1}^{n}\left(L_{k+1}^{-}\right)^{-j}|e_i(t_k+s)|\right)^2\right)^{1/2} \\
& \leqslant \left(\sum_{j=1}^{n} c^2(t_k+s) n\left\|\left(\mathcal{L}_{k+1}^{-}\right)^{-1} e(t_k+s)\right\|^2\right)^{1/2} \\
& = n\, c(t_k+s)\left\|\left(\mathcal{L}_{k+1}^{-}\right)^{-1} e(t_k+s)\right\|. \qquad (19)
\end{aligned}
$$

From formula (15) and inequality (19), we get

$$
\begin{aligned}
\|R\| \leqslant & \|I + \alpha KC\| \int_0^{\delta_k}\exp\left(\|A\| L_{k+1}^{-}(\delta_k - s)\right)\ldots \\
& n\, c(t_k+s)\left\|\left(\mathcal{L}_{k+1}^{-}\right)^{-1} e(t_k+s))\right\| ds \qquad (20)
\end{aligned}
$$

We have for all s in $[t_k, t_{k+1})$

$$
\begin{aligned}
\left(\mathcal{L}_{k+1}^{-}\right)^{-1}\dot{e}(t_k+s) = & L_{k+1}^{-} A\left(\mathcal{L}_{k+1}^{-}\right)^{-1} e(t_k+s) \\
& + \left(\mathcal{L}_{k+1}^{-}\right)^{-1}\Delta(t_k+s).
\end{aligned}
$$

Denoting by $w(s)$ the expression $\left(\mathcal{L}_{k+1}^{-}\right)^{-1} e(t_k+s)$, this gives

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\|w(s)\| = & \frac{\langle \dot{w}(s), w(s)\rangle}{\|w(s)\|} \\
& \leqslant \left\|L_{k+1}^{-} A\left(\mathcal{L}_{k+1}^{-}\right)^{-1} e(t_k+s)\right\| \\
& + \left\|\left(\mathcal{L}_{k+1}^{-}\right)^{-1}\Delta(\hat{x}(t_k+s), e(t_k+s), u(t_k+s))\right\| \\
& \leqslant \left(L_{k+1}^{-}\|A\| + c(t_k+s)\right)\left\|\left(\mathcal{L}_{k+1}^{-}\right)^{-1} e(t_k+s)\right\|.
\end{aligned}
$$

Hence, we finally obtain

$$\left\| \left(\mathcal{L}_{k+1}^{-}\right)^{-1} e(t_k+s) \right\| \leqslant \exp\left(\int_0^s L_{k+1}^{-} \|A\| + nc(t_k+r)dr \right) \left\| \left(\mathcal{L}_{k+1}^{-}\right)^{-1} e_k \right\|. \tag{21}$$

Consequently, according to (20) and (21), we get

$$\begin{aligned} \|R\| &\leqslant \|I+\alpha KC\| \int_0^{\delta_k} \exp\left(\|A\| L_{k+1}^{-}(\delta_k - s)\right) nc(t_k+s) \ldots \\ &\quad \exp\left(\int_0^s L_{k+1}^{-}\|A\| + nc(t_k+r)\mathrm{d}r \right) \| \left(\mathcal{L}_{k+1}^{-}\right)^{-1} e_k \| \mathrm{d}s \\ &= \|I+\alpha KC\| \exp\left(\|A\|\alpha\right) \left\| \left(\mathcal{L}_{k+1}^{-}\right)^{-1} e_k \right\| \times \\ &\quad \int_0^{\delta_k} nc(t_k+s) \exp\left(\int_0^s nc(t_k+r)\mathrm{d}r \right) \mathrm{d}s \\ &= \|I+\alpha KC\| \exp\left(\|A\|\alpha\right) \left\| \left(\mathcal{L}_{k+1}^{-}\right)^{-1} e_k \right\| \times \\ &\quad \left[\exp\left(n \int_0^{\delta_k} c(t_k+r)dr \right) - 1 \right]. \end{aligned}$$

Hence, employing Lemma 6 this gives the existence of two continuous function N_1 and N_2 such that

$$\begin{aligned} &2E_k'\left(\mathcal{L}_{k+1}^{-}\right)^{-1} \mathcal{L}_k Q(\alpha)' \Psi P \Psi R \\ &\leqslant \left\| \left(\mathcal{L}_{k+1}^{-}\right)^{-1} e_k \right\|^2 N_1(\alpha) \left[\exp\left(n \int_0^{\delta_k} c(t_k+r)dr \right) - 1 \right]. \end{aligned}$$

with

$$N_1(\alpha) = 2\|Q(\alpha)\| \, \|I+\alpha KC\| \, \exp(\|A\|\alpha) \, \frac{\|P\|}{(1-a_1\alpha)^{2n}}$$

Moreover,

$$\begin{aligned} &R'\Psi P \Psi R \leqslant \\ &\left\| \left(\mathcal{L}_{k+1}^{-}\right)^{-1} e_k \right\|^2 N_2(\alpha) \left[\exp\left(n \int_0^{\delta_k} c(t_k+r)dr \right) - 1 \right]^2. \end{aligned}$$

where

$$N_2(\alpha) = \|I+\alpha KC\|^2 \exp(2\|A\|\alpha) \, \frac{\|P\|}{(1-a_1\alpha)^{2n}}$$

The two previous inequalities imply that (18) holds. $\square$

Step 3: Lyapunov analysis
With the two bounds obtained for T_1 and T_2 in Lemmas 3 and 5 , we finally get

$$\begin{aligned} V(E_{k+1}) - V(E_k) &\leqslant \left\| \left(\mathcal{L}_{k+1}^{-}\right)^{-1} e_k \right\|^2 \cdot \Big[N_1(\alpha)[e^{\beta}-1] \\ &+ N_2(\alpha)[e^{\beta}-1]^2 - \frac{a_2 p_3 p_1}{a_3} \left[e^{\frac{a_3}{n}\beta} - 1 \right] - \frac{\alpha p_1}{4p_2} \Big], \end{aligned}$$

where $\beta = n \int_0^{\delta_k} c(t_k+r)dr$. Note that for all α, thanks to a good choice of a_3 and a_2 it yields that the right-hand member in the previous inequality is negative for every β. For example, if we take $a_3 = 2n$, the previous inequality becomes

$$\begin{aligned} V(E_{k+1}) - V(E_k) &\leqslant \left\| \left(\mathcal{L}_{k+1}^{-}\right)^{-1} e_k \right\|^2 \cdot \Big[-\frac{\alpha p_1}{4p_2} \\ &+ \left[e^{\beta}-1\right] \left[-\frac{a_2 p_3 p_1}{2n} \left[e^{\beta}+1\right] + N_1(\alpha) + N_2(\alpha)[e^{\beta}-1] \right] \Big] \\ &\leqslant \left\| \left(\mathcal{L}_{k+1}^{-}\right)^{-1} e_k \right\|^2 \cdot \Big[-\frac{\alpha p_1}{4p_2} \\ &+ \left[e^{\beta}-1\right] \left[e^{\beta}+1\right] \left[-\frac{a_2 p_3 p_1}{2n} + N_1(\alpha) + N_2(\alpha) \right] \Big]. \end{aligned}$$

If $a_2 \geqslant 2n \frac{N_1(\alpha)+N_2(\alpha)}{p_3 p_1}$ it yields

$$V(E_{k+1}) - V(E_k) \leqslant -\frac{\alpha p_1}{4p_2} \left\| \left(\mathcal{L}_{k+1}^{-}\right)^{-1} e_k \right\|^2.$$

The function V being positive definite, it yields that

$$\lim_{k\to+\infty} \left\| \left(\mathcal{L}_{k+1}^{-}\right)^{-1} e_k \right\| = 0 \, .$$

The function L being upper and lower bounded (by Lemma 1), this implies that the error e_k goes to zero. With (11), we get the result. $\square$

Remark 1 *An interesting question would be to find the optimal value of α in $(0, \alpha_m)$ to maximize the measurement step-size. This is a difficult question that requires some further analysis and depends on the bound on $L(\cdot)$. Indeed, from the equality $\delta_k = \frac{\alpha}{L(t_k^-)}$, we wish to select α large. However, at the same time, a large α implies a large parameter a_3 which implies also large $L(t_k^-)$. Hence, a nonlinear optimization has to be carried out.*

4 Illustrative example

In this section, the performance of the proposed observer is illustrated through a bioreactor. In most cases, a cheap and reliable instrumentation required for real-time measurement of key variables of such process (biomass, substrate) are not available. Nevertheless, biomass measurement can be obtained using off-line analysis (sampled measurements) which requires time and staff investment. The proposed approach allows to reduce the measurements frequency and consequently, the monitoring cost is also reduced.

The bioprocess considered is an academic bioreactor which consists of a microbial culture which involves a biomass X growing on a substrate S. The bioprocess is supposed to be continuous with a scalar dilution rate D and an input substrate concentration S_{in} (which is assumed to be constant). Under these conditions and using the Contois model, the dynamical model of the process is given by

$$\begin{cases} \dot{X} = \dfrac{\mathcal{K}_1 S}{\mathcal{K}_2 X + S} X - DX \\ \dot{S} = -\mathcal{K}_3 \dfrac{\mathcal{K}_1 S}{\mathcal{K}_2 X + S} X - D\,(S - S_{in}), \end{cases} \tag{22}$$

where the $\mathcal{K}_i$'s (i=1,2,3) are positive constants. Our objective is the on-line estimation of the substrate concentrations S through sampled biomass measurements. In the case where the output is assumed to be time-continuous, the authors in [9] gave a stationary high gain observer. In the sequel, the same hypothesis as in [9] and the same notations are used.
Set the state vector $z = [X, S]'$, the input $u = D$ and the output $y(t_k) = X(t_k)$. Under the constraint $0 < u_{\min} \leqslant u \leqslant u_{\max} < \mathcal{K}_1$, the authors in [9], determined a compact domain $\mathcal{M}_z \in \mathbb{R}^2$ which is invariant under the normal form (1). In the sequel, we choose $\mathcal{K}_3 = 1$ which means that there is no change of volume when the substrate transforms into microorganisms, the two other values being $\mathcal{K}_1 = \mathcal{K}_2 = 1$ (notice that $\mathcal{K}_1 = 1$ up a change of time unit)

$$\mathcal{M}_z = \{z \in \mathbb{R}^2 : X \geqslant \epsilon_1, S \geqslant \epsilon_2, X + S \leqslant 1\},$$

where $\epsilon_1 = \frac{(1-u_{\max})\epsilon_2}{S_{in} u_{\max}}$. Then, using the change of coordinates $z \in \mathcal{M}_z \mapsto x = \Phi(z)$, defined as

$$\Phi(z) = \left[X, \frac{SX}{X+S}\right]',$$

system (22) takes the normal form [3] (1) with $n = 2$, and

$$f_1(x, u) = -u x_1,$$
$$f_2(x, u) = S_{in} u - \left(1 + u + \frac{2 S_{in} u}{x_1}\right) x_2 + \left(\frac{2-u}{x_1} + \frac{S_{in} u}{x_1^2}\right) x_2^2,$$

and x evolving in $\mathcal{M}_x = \Phi(\mathcal{M}_z)$.

Moreover, the function f_2 can be easily extended to a global Lipschitz C^1 function on the whole domain $\mathbb{R}^2 \times \mathcal{M}_u$. For all $(\hat{x}, x, u) \in \mathbb{R}^2 \times \mathcal{M}_z \times \mathcal{M}_u$, where $\mathcal{M}_u = [u_{\min}, u_{\max}]$, we can write

$$|f_1(x,u) - f_1(\hat{x},u)| \leqslant c_{11}(u)|x_1 - \hat{x}_1|$$
$$\begin{aligned} |f_2(x,u) - f_2(\hat{x},u)| &\leqslant |f_2(x_1,x_2,u) - f_2(x_1,\hat{x}_2,u)| \\ &\quad + |f_2(x_1,\hat{x}_2,u) - f_2(\hat{x}_1,\hat{x}_2,u)| \\ &\leqslant c_{21}(x_1,x_2,\hat{x}_2,u)|x_2 - \hat{x}_2| \\ &\quad + c_{22}(x_1,\hat{x}_1,x_2,u)|x_1 - \hat{x}_1|, \end{aligned}$$

where

$$c_{11}(u) = -u,$$
$$c_{21}(x,\hat{x},u) = -\left(1 + u + \frac{2S_{in}u}{x_1}\right) + \left(\frac{2-u}{x_1} + \frac{S_{in}u}{x_1^2}\right)(x_2 + \hat{x}_2),$$

and

$$c_{22}(x,\hat{x},u) = \frac{2S_{in}u\hat{x}_2 + (2-u)\hat{x}_2^2}{\hat{x}_1 x_1} - S_{in}u\hat{x}_2^2 \frac{\hat{x}_1 + x_1}{\hat{x}_1^2 x_1^2}.$$

Setting

$$c(\hat{x},u) = \max_{x \in \mathcal{M}_x} \{c_{11}(u), c_{21}(x,\hat{x},u), c_{22}(x,\hat{x},u)\},$$

we obtain

$$|f_1(x,u) - f_1(x+e,u)| \leqslant c(\hat{x},u)|e_1|\ ,$$
$$|f_2(x,u) - f_2(x+e,u)| \leqslant c(\hat{x},u)[|e_1| + |e_2|]\ ,$$

Now, it suffices to use (5)-(7) to give the updated sampling time observer. The observer parameters have been selected through a trial and error procedure as follows:

$$K = [-2, 1]',\ \alpha = .9\ ,\ a_1 = 1\ ,\ a_2 = .1\ ,\ a_3 = .1\ .$$

4.1 Simulation results

For the simulation test [4], the output has been corrupted by an additive noisy signal as shown in Figure 1. The observer simulation was performed under similar operating conditions as the model ($\mathcal{K}_i = 1$) and $S_{in} = 0.1$, and $u : [0, 40] \to \mathbb{R}$ is displayed in Figure 1.

Figure 2 displays the calculated values of the sampling-time δ_k. It may be noted that the sampling-time suggested by the proposed approach is relatively small when the estimated dynamics speed is important and take a large value when the dynamics speed is close to zero. This behavior is quite natural: when the system is not much excited, the state variables vary slowly and we can wait a little bit more time between two measurements; moreover, the designed gain L of the observer can be chosen small.

[3] Notice that system (22) can be put in normal form whatever the values of $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{K}_3$.

[4] The Matlab files can be downloaded from `https://sites.google.com/site/vincentandrieu/`

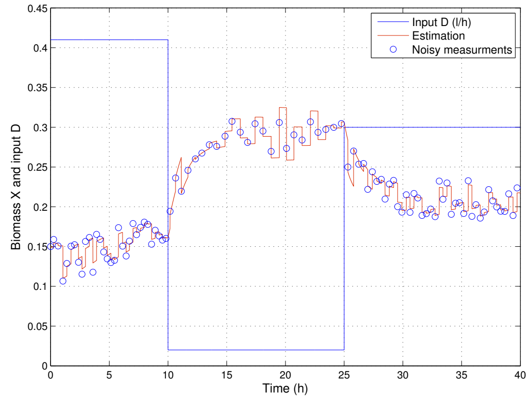

Fig. 1. Input $u = D$ and output $y(t_k) = X(t_k)$ with measurement noise.

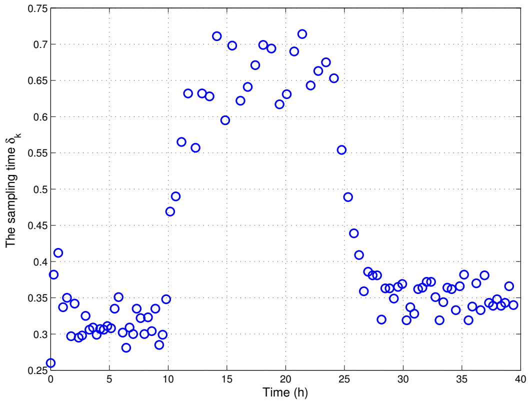

Fig. 2. Updated sampling time δ_k.

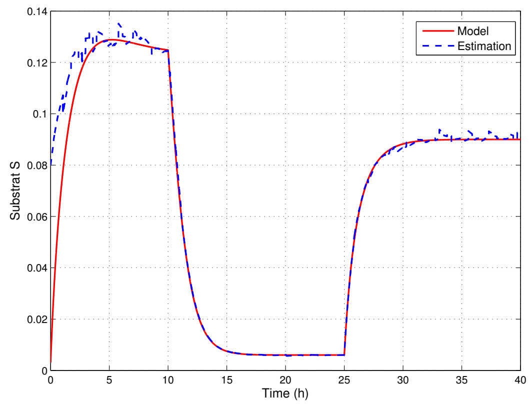

Fig. 3. S given by the model (22) compared to $\hat{S}$ given by system (5)-(7).

Figure 3 illustrates the impact of the measurement noise on the observer performances. We can see that the observer behavior with respect to the measurement noise is satisfactory.

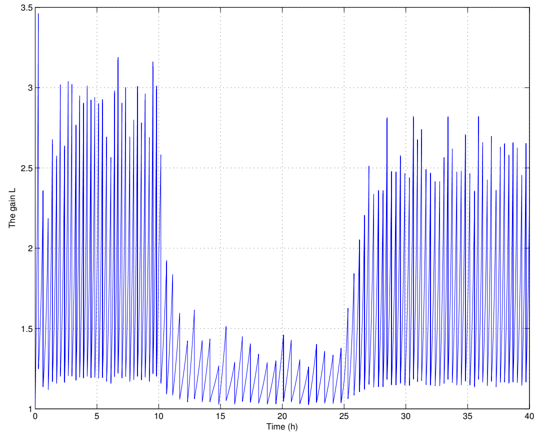

Fig. 4. Evolution of L .

5 Conclusion

In this paper, a high gain observer for continuous-discrete time systems in the observability normal form has been designed. The problem of observer synthesis for these systems is related to the sampling time of the output measurement which is always uniform and should be small to guarantee the observer convergence. To overcome this constraint which increases the control cost, a high gain updated sampling-time observer has been proposed. The principal advantage of this observer is that it requires the less knowledge as possible from the output measurement. The obtained results have been illustrated in the biological process and demonstrated good performances.

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A Proofs of Lemmas

A.1 Proof of Lemma 1

Assuming that the input u is an essentially bounded time function (with unknown bound), thanks to (4) we get that the function $t \mapsto c(\hat{x}(t), u(t))$ is essentially upper bounded on the time of existence of the solution. Let c_m be an essential upper bound of $c(\hat{x}(t), u(t))$. Note that by integrating equation (6b) with the previous upper bound on the interval $[t_k, t_{k+1})$, it yields $M(t) \leqslant e^{a3c_m(t-t_k)}$ for every $t \in [t_k, t_{k+1})$, reporting this inequality in (6a)it yields for all k and $t \in [t_k, t_{k+1})$

$$L(t) \leqslant \kappa(t - t_k)L(t_k), \text{for every } t \in [t_k, t_{k+1}) \qquad \text{(A.1)}$$

where κ is an increasing function such that $\kappa(0) = 1$ defined as

$$\kappa(s) = \exp(a_2 c_m s \exp(a_3 c_m s)).$$

Hence, from (6c), and (A.1) with $t = t_{k+1}^-$, we get

$$L(t_{k+1}) \leqslant (1 - a_1\alpha)\kappa(\delta_k)L(t_k) + a_1\alpha \ .$$

Note moreover that we have $\dot{L}(s) \geqslant 0$ for all s in $\left[t_k, t_{k+1}^-\right)$, and so $L(t_{k+1}^-) \geqslant L(t_k)$. Hence, since we have $L(t_{k+1}^-)\delta_k = \alpha$ we get $\delta_k \leqslant \frac{\alpha}{L(t_k)}$ which gives

$$\frac{L(t_{k+1})}{L(t_k)} \leqslant (1 - a_1\alpha)\kappa\left(\frac{\alpha}{L(t_k)}\right) + \frac{a_1\alpha}{L(t_k)}. \qquad \text{(A.2)}$$

To see that the sequence $(L(t_k)))_{k\geqslant 0}$ is bounded, let us introduce φ the function defined on the interval $(0, +\infty)$ as

$$\varphi(\ell) = (1 - a_1\alpha)\kappa\left(\frac{\alpha}{\ell}\right) + \frac{a_1\alpha}{\ell}.$$

Notice that φ is decreasing on this interval, that $\lim_{\ell\to 0}\varphi(\ell) = +\infty$ and that $\lim_{\ell\to+\infty}\varphi(\ell) = 1 - a_1\alpha < 1$; so there exists a unique $\ell_1 \in (0, +\infty)$ such that $\varphi(\ell_1) = 1$. Assume now that $L(t_k) \leqslant \ell_1$ for every $k \geqslant 0$, then we can say that the sequence $(L(t_k))_{k\geqslant 0}$ is bounded. If $L(t_k) \geqslant \ell_1$ for every $k \geqslant 0$, the inequality (A.2) implies that

$$\begin{aligned} L(t_{k+1}) &\leqslant L(t_k)\varphi(L(t_k)) \\ &\leqslant L(t_k)\varphi(\ell_1) \qquad \text{(because } L(t_k) \geqslant \ell_1) \\ &= L(t_k) \end{aligned}$$

and, arguing by induction, we easily see that $L(t_k) \leqslant L(t_0)$ for every k. The last situation is when some $L(t_k)$ are less than ℓ_1 for $k \in \{1, \ldots, k_0\}$ and becomes greater than ℓ_1 for $k \geqslant k_0 + 1$. So assume that we have, for some index k_0,

$$L(t_{k_0}) \leqslant \ell_1 \qquad L(t_{k_0+i}) > \ell_1 \text{for} i = 1, \ldots, N$$

with, possibly, $N = +\infty$. As above, we can prove that $L(t_{k_0+i}) \leqslant \ldots \leqslant L(t_{k_0+1})$. Now, from (A.2) and taking into account the definition of φ, we get

$$\begin{aligned} L(t_{k_0+1}) &\leqslant L(t_{k_0})\varphi(L(t_{k_0})) \\ &\leqslant \ell_1\varphi(1) \qquad\qquad \text{as } 1 \leqslant L(t_{k_0}) \leqslant \ell_1. \end{aligned}$$

Thus, we proved that $L(t_k) \leqslant \max(\ell_1, \ell_1\varphi(1))$ for every index k. Finally, the boundedness of the sequence $(L(t_k))_{k\geqslant 0}$ and inequality (A.1) imply that the function $t \mapsto L(t)$ is bounded on $\mathbb{R}+$.

Notice that the equality $L(t_{k+1}^-)\delta_k = \alpha$ and the boundedness of $L(t)$ imply that δ_k is bounded from below and so the sampling time cannot tend to zero.

A.2 Proof of Lemma 4

In order to prove Lemma 4, we need the following lemma which will be proved in the next section.

Lemma 6 *The matrix P satisfies the following property for all a_1 and α such that $a_1\alpha < 1$*

$$\Psi P \Psi \leqslant \psi_0(\alpha) P \psi_0(\alpha) \ , \tag{A.3}$$

where

$$\psi_0(\alpha) = \operatorname{diag}\left(\frac{1}{1-a_1\alpha}, \ldots, \frac{1}{(1-a_1\alpha)^n}\right).$$

Given v in $S^{n-1} = \{v \in \mathbb{R}^n \mid \|v\| = 1\}$, consider the function

$$\nu(\alpha, v) = v' Q(\alpha)' \psi_0(\alpha) P \psi_0(\alpha) Q(\alpha) v \ .$$

We have

$$\begin{aligned} &\nu(0,v) = v'Pv, \\ &\frac{\partial \nu}{\partial \alpha}(0,v) = v'\left[P[A + KC + a_1 D] + [A + KC + a_1 D]'P\right] v, \end{aligned}$$

so using the inequalities in (9) and setting $a_1 = \frac{1}{2p_2p_4}$, we get

$$\begin{aligned} \frac{\partial \nu}{\partial \alpha}(0,v) &\leqslant v'\left(a_1 p_4 P - \frac{1}{p_2}P\right) v \\ &= -\frac{1}{2p_2}\, v'Pv. \end{aligned} \tag{A.4}$$

Now, we can write

$$\nu(\alpha, v) = v'Pv + \alpha\frac{\partial \nu}{\partial \alpha}(0,v) + \rho(\alpha, v)$$

with $\lim_{\alpha\to 0} \frac{\rho(\alpha,v)}{\alpha} = 0$. This equality together with (A.4) imply that

$$\nu(\alpha, v) \leqslant v'Pv\left[1 - \alpha\frac{1}{2p_2}\right] + \rho(\alpha, v).$$

The vector v being in a compact set and the function r being continuous, there exists α_m such that for all α in $[0, \alpha_m)$ we have $r(\alpha, v) \leqslant \alpha\frac{1}{4p_2}v'Pv$ for all v. This gives

$$\nu(\alpha, v) \leqslant v'Pv\left[1 - \alpha\frac{1}{4p_2}\right] , \forall\, \alpha \in [0, \alpha_m) \, , \forall\, v \in S^{n-1} \, .$$

This property being true for every v, this ends the proof of Lemma 4.

A.3 Proof of Lemma 6

Consider the matrix function defined as

$$\mathcal{P}(s) = \operatorname{diag}(s, \ldots, s^n) P \operatorname{diag}(s, \ldots, s^n) \ .$$

Note that for all v in $\mathbb{R}^n$

$$\begin{aligned} &\tfrac{d}{ds} v'\mathcal{P}(s)v \\ &= \tfrac{1}{s} v' \operatorname{diag}(s, \ldots, s^n)(D'P + PD) \operatorname{diag}(s, \ldots, s^n) v > 0. \end{aligned}$$

Hence, $\mathcal{P}$ is an increasing function. Furthermore, we have

$$\begin{aligned} \Psi P \Psi &= \mathcal{L}_{k+1}^{-1}\mathcal{L}_{k+1}^{-} P \mathcal{L}_{k+1}^{-}\mathcal{L}_{k+1}^{-1} \\ &= \mathcal{P}\left(\frac{L_{k+1}^-}{L_{k+1}^-(1-a_1\alpha) + a_1\alpha}\right), \end{aligned}$$

so as

$$\frac{L_{k+1}^-}{L_{k+1}^-(1-a_1\alpha) + a_1\alpha} \leqslant \frac{1}{1-a_1\alpha},$$

we get the inequality of Lemma 6

$$\Psi P \Psi \leqslant \mathcal{P}\left(\frac{1}{1-a_1\alpha}\right) \ .$$